JOURNAL OF APPROXIMATION THEORY 13, 443–450 (1975)

Korovkin Theorems for Positive Linear Operators

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Communicated by P. L. Butzer

DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

INTRODUCTION

The well-known theorem of Korovkin [3, 4] states that a sequence (T_n) of positive endomorphisms on $\mathscr{C}([0, 1])$ converges to the identity operator provided that $\lim_{n\to\infty} T_n(g^i) = g^i$ for $i = 0, 1, 2$, where $g \in \mathscr{C}([0, 1])$ denotes the identity mapping. In recent years many generalizations of this result have been given. The concept of a Korovkin closure (or shadow) turned out to be an adequate tool for the solution of many problems arising in the context of Korovkin theorems.

If E and F are topological vector lattices in the terminology of Peressini [7] or Schaefer [8] and if H is a subset of E, the Korovkin closure of H with respect to a continuous linear lattice homomorphism $S : E \to F$ is the set of all $x \in E$ satisfying the following condition:

For each net $(T_i)_{i\in I}$ of positive linear operators, $(T_i(x))_{i\in I}$ converges to $S(x)$ provided that $\lim_{i\in I} T_i(y) = S(y)$ for all $y \in H$.

We here also deal with sequences of positive linear operators as well as with nets (and sequences) of continuous positive linear operators of E into F. Since the Korovkin closures are clearly subspaces of E, w.l.o.g., H may be assumed to be a vector subspace of E.

Based on a preceding publication [2] in this note a complete and simple characterization of shadows is given for nets of positive and continuous positive linear maps. Concerning sequences Korovkin closures are characterized at least in the case where F is metrizable and H has a countable algebraic basis.

While in [2] H was assumed to be cofinal in E, we do not need any additional assumption on H here. In fact, it turns out that, w.l.o.g., H may be assumed to be cofinal in E. More precisely, the shadow is contained in the linear lattice ideal generated by H.

In contrast to [2] we here use a lemma on uniqueness of extensions of isotone maps in order to obtain the description of Korovkin closures by so-called (H, S)-affine elements. Surprisingly, this lemma is the common background of many Korovkin theorems for positive linear operators.

Notations and Definitions

(a) Let H be a subset of some set E and let F be a Hausdorff topological space. Suppose that $\mathscr{T}$ is a class[1] of nets of mappings from E into F. Given a map $S : E \to F$, the *Korovkin closure* or *shadow*[2] $\overline{H}_S^{\mathscr{T}}$ of H with respect to $\mathscr{T}$ and S is the set of all $x \in E$ satisfying the following condition:

For each net $(T_i)_{i\in I} \in \mathscr{T}$ such that $\lim_{i\in I} T_i(y) = S(y)$ for all $y \in H$ $(T_i(x))_{i\in I}$ converges to $S(x)$.

(b) We do not try to attain utmost generality. Hence E and F will always assumed to be *Hausdorff topological vector lattices.* Moreover, we will only deal with subclasses of *the class $\mathscr{P}$ of all nets of positive linear operators* of E into F. Finally, throughout this note, $S : E \to F$ will always denote a *continuous linear lattice homomorphism* and H will be a *linear subspace* of E.

(c) For each subclass $\mathscr{T}$ of $\mathscr{P}$, $\mathscr{T}'$ denotes the subset of all sequences in $\mathscr{T}$. $\mathscr{P}_c$ is the subclass of $\mathscr{P}$ consisting of all nets of continuous positive linear operators.

(d) For each $x \in E$, we define $H_x := \{y \in H : y \geqslant x\}$ and

$$H^x := \{y \in H : y \leqslant x\}$$

(note that H_x and H^x may be empty). Furthermore,

$$\hat{H}_x := \{\inf A : \varnothing \neq A \subset H_x , A \text{ finite}\},$$

$\check{H}^x := \{\sup B : \varnothing \neq B \subset H^x, B \text{ finite}\}$. Obviously, $\hat{H}_x$ is downward directed, while $\check{H}^x$ is upward directed. An element $x \in E$ is called *(H, S)-affine* iff $H_x \neq \varnothing$, $H^x \neq \varnothing$ and $\lim_{y\in\hat{H}_x} S(y) = S(x) = \lim_{y\in\check{H}^x} S(y)$. The subset of all (H, S)-affine elements in E will be denoted by $\mathscr{A}_S(H)$. The fundamental importance of (H, S)-affine elements will become clear in the next section.

(e) For any set M $\mathfrak{P}_e(M)$ denotes the system of all finite nonempty subsets of M. Note that $\mathfrak{P}_e(M)$ is upward directed by inclusion!

[1] If filters are used instead of nets, the introduction of classes can be avoided. Since, however, the limit statements are easier to formulate with nets then with filters, we prefered to use nets in the definition of Korovkin closures.

[2] Korovkin closures and shadows are distinct objects in [2]!

1. Uniqueness of Extensions of Isotone Maps

LEMMA 1.1. *Let M, N be two lattices and let G be a subset of E. If T, Q are two isotone maps of M into N coinciding on G, then $T = Q$ on*

$$\mathscr{G} := \{x \in M : \inf\{T(y) : y \in G, y \geqslant x\} = T(x) = \sup\{T(z) : z \in G, z \leqslant x\}\}.$$

Proof. Let $x \in \mathscr{G}$. Since

$$T(y) = Q(y) \geqslant \sup(T(x), Q(x)) \geqslant \inf(T(x), Q(x)) \geqslant T(z) = Q(z)$$

for all $y, z \in G$ satisfying $y \geqslant x \geqslant z$, it follows that

$$\sup(T(x), Q(x)) = T(x) = Q(x) = \inf(T(x), Q(x)).$$

COROLLARY 1.2. *Each (H, S)-affine element of E is contained in $\overline{H}_S^{\mathscr{P}}[2]$.*

Proof[3]. Let I be a directedly ordered set. If

$$K_0 := \{(x_i) \in F^I : \lim_{i \in I} x_i = 0\},$$

then K_0 is an order-convex[4] linear sublattice of the product lattice F^I. Hence $N := F^I/K_0$ is a lattice [5, p. 49]. Let $q : F^I \to N$ be the quotient map. Given a net $(T_i)_{i \in I} \in \mathscr{P}$ (with index set I) we define $\tilde{S}$, $\tilde{T} : E \to N$ by setting

$$\tilde{T}(x) = q(T_i(x))_{i \in I}) \text{ and } \tilde{S}(x) = q((S_i(x))_{i \in I}), \text{ where } S_i := S \text{ for all } i \in I.$$

Note that, for any $x \in E$,

$$\tilde{T}(x) = \tilde{S}(x) \text{ iff } \lim_{i \in I} T_i(x) = S(x).$$

Hence we derive from 1.1 :

$$\{x \in E : \inf \tilde{S}(H_x) = \inf \tilde{S}(\hat{H}_x) = \tilde{S}(x) = \sup \tilde{S}(\underset{\vee}{H}^x) = \sup \tilde{S}(H^x)\} \subset \overline{H}_S^{\mathscr{P}}.$$

The proof will be complete, if we can show that $\mathscr{A}_S(H)$ is contained in the set on the left-hand side of the inclusion. To do this, let $x \in \mathscr{A}_S(H)$, and suppose that $(z_i) \in F^I$ satisfies $\tilde{S}(y) \geqslant q((z_i)) \geqslant \tilde{S}(x)$ for all $y \in \hat{H}_x$. For each $y \in \hat{H}_x$ there is a net $(e_{i,y})_{i \in I} \in K_0$ such that $S(y) \geqslant z_i + e_{i,y} \geqslant S(x)$ for all $i \in I$. Consequently, if U, V are solid zero-neighborhoods in F satisfying $V + V \subset U$, $V = -V$, then there is an element $y \in \hat{H}_x$ and an index $i_0 \in I$ such that

[3] The construction used in this proof was first applied to Korovkin theorems by Scheffold [10, 9].

[4] A subset W of an ordered set U is called order-convex (see [5] for the terminology) iff $x \in U$ and $y \leqslant x \leqslant z$ for some $y, z \in W$ implies $x \in W$.

$S(y) \in V + S(x)$ (since $x \in \mathscr{A}_S(H)$) and $e_{i,y} \in V$ for all $i \geqslant i_0$. But this implies that $z_i \in S(x) + U$ for all $i \geqslant i_0$, or, since U was arbitrary, $\tilde{S}(x) = q((z_i))$. Hence, $\inf_{y \in \hat{H}_x} \tilde{S}(y) = \tilde{S}(x)$. Similarly, one proves that $\sup_{y \in \underline{H}^x} \tilde{S}(y) = \tilde{S}(x)$.

Remarks 1.3. Note that the proof of 1.2 remains valid if the hypotheses on E, F, S and $\mathscr{P}$ are weakened as follows:

(i) E may be assumed to be a lattice only, F to be a locally solid lattice, H to be a subset of E. The class $\mathscr{P}$ can be replaced by the class $\mathscr{F}$ of all nets of isotone maps of E into F. Moreover, $S : E \to F$ need only be a lattice homomorphism (see also [2]).

(ii) The proof of 1.2 still holds if we replace the class $\mathscr{P}$ by the class of all nets (T_i) of linear operators from E into F such that

$$\lim_{i \in I} (T_i(x))^- = 0 \qquad \text{for all} \quad x \geqslant 0 \ (x \in E).$$

Indeed, the mapping $\tilde{T}$ defined in 1.2 remains positive under this assumption.

(iii) Still another generalization of the corollary can be obtained by weakening the topological hypotheses on F. It is not hard to show that one can derive Korovkin theorems with respect to order-convergence (as for instance convergence almost everywhere on function spaces) and relatively uniform convergence from Lemma 1.1 by similar methods as in 1.2 [1, 2, 11].

For reasons of conciseness we shall stick to our initial assumptions in Notations and Definitions neglecting the various modifications of Korovkin type theorems mentioned above. Since $\overline{H}_S^{\mathscr{T}_2} \subset \overline{H}_S^{\mathscr{T}_1}$ for any two subclasses $\mathscr{T}_1, \mathscr{T}_2 \subset \mathscr{P}$ satisfying $\mathscr{T}_1 \subset \mathscr{T}_2$ we conclude from 1.2:

$$\mathscr{A}_S(H) \subset \overline{H}_S^{\mathscr{P}} \subset \overline{H}_S^{\mathscr{P}'} \subset \overline{H}_S^{\mathscr{P}'_c} \qquad \text{and} \quad \mathscr{A}_S(H) \subset \overline{H}_S^{\mathscr{P}} \subset \overline{H}_S^{\mathscr{P}_c} \subset \overline{H}_S^{\mathscr{P}'_c}.$$

In the next section we shall see that we do have equality in many cases of practical interest.

2. Equivalent Characterizations of Shadows

The general assumptions for this section are those of Notations and Definitions.

THEOREM 2.1. *If the positive linear forms on E separate points, then* $\mathscr{A}_S(H) = \overline{H}_S^{\mathscr{P}}$. *Moreover, if H has a countable algebraic basis and the topology on F is metrizable, then the* (H, S)*-affine elements also coincide with the sequential Korovkin closure* $\overline{H}_S^{\mathscr{P}'}$. *The same is true for* $\overline{H}_S^{\mathscr{P}_c}$ *(resp.* $\overline{H}_S^{\mathscr{P}_c'}$*) if the positive continuous linear forms on E separate points.*

Proof. (a) Let us first show that the respective shadows are contained in $H_0 := \{x \in E : H_x \neq \varnothing \neq H^x\}$. To prove this, let $x \in E \setminus H_0$. For each $A \in \mathfrak{P}_e(H)$ define A^c to be the convex hull of $\{|A|^2 \cdot a : a \in A\}$, where $|A|$ is the cardinal number of A. Since $x \notin H_0$, either x or $-x$ is not contained in $A^c - E_+$. W.l.o.g. we may assume that $x \notin A^c - E_+$ (otherwise we replace x by $-x$). From the finiteness of A it now follows immediately that A^c is $\sigma(E, E^*)$-compact, where E^* is the algebraic dual of E. Moreover, E_+ is $\sigma(E, E^*)$-closed since the positive linear forms on E separate points (for a proof, see [2, 4.5]), and so is $A^c - E_+$. Consequently, there is a convex $\sigma(E, E^*)$-neighborhood U of x such that $U \cap (A^c - E_+) = \varnothing$. A well-known separation theorem [5, p. 82] now yields the existence of a positive linear form ω_A on E satisfying $0 \neq \omega_A(x) > \sup \omega_A(A^c - E_+)$. Define $T_A : E \to F$ by setting

$$T_A(z) = S(z) + |A| \cdot \frac{\omega_A(z)}{\omega_A(x)} \cdot y_0, \qquad \text{where} \quad y_0 \in F_+ \setminus \{0\} \text{ is fixed.}$$

Then we obtain $\lim_{A \in \mathfrak{P}_e(H)} T_A(y) = S(y)$ for all $y \in H$.

Indeed, if $A \in \mathfrak{P}_e(H)$ is such that $y, -y \in A$, it follows that

$$S(y) - \frac{1}{|A|} \cdot y_0 \leqslant S(y) - \frac{\omega_A(|A|^2 \cdot (-y))}{|A| \cdot \omega_A(x)} \cdot y_0 = T_A(y)$$
$$= S(y) + \frac{\omega_A(|A|^2 \cdot y)}{|A| \cdot \omega_A(x)} \cdot y_0 \leqslant S(y) + \frac{1}{|A|} \cdot y_0 .$$

On the other hand, $(T_A(x))_{A \in \mathfrak{P}_e(H)}$ is divergent. Hence $x \notin \overline{H}_S^{\mathscr{P}}$.

If the continuous positive linear forms on E separate points it is possible to substitute the topology $\sigma(E, E^*)$ by $\sigma(E, E')$, where E' is the topological dual of E. It follows that ω_A and hence T_A will be continuous. Consequently, $x \notin H_S^{\mathscr{P}_c}$ in this case.

Finally, if H has a countable algebraic basis B, then we may use the countable subnet $(T_A)_{A \in \mathfrak{P}_e(B)}$, which can be rewritten as a sequence, to prove the assertion for the sequential Korovkin closures $\overline{H}_S^{\mathscr{P}_c'}$ and $\overline{H}_S^{\mathscr{P}'}$, respectively.

(b) Thus, to complete the proof, it suffices to show that each element $x \in H_0$ which is contained in the Korovkin closure $\overline{H}_S^{\mathscr{P}}$ (resp. $\overline{H}_S^{\mathscr{P}'}, \overline{H}_S^{\mathscr{P}_c'}, \overline{H}_S^{\mathscr{P}_c'}$) is (H, S)-affine. Since this is obviously true for $x \in H$, we may assume that $x \in (H_0 \cap \overline{H}_S^{\mathscr{P}}) \setminus H$. Choose an element $y_0 \in H^x$, and note that $0 \neq x - y_0 \in \overline{H}_S^{\mathscr{P}} \cap E_+$. Using a result in [2, 4.3], there is a net $(L_i)_{i \in I}$ of positive endomorphisms on E (resp. a sequence of positive endomorphisms, a net of continuous positive endomorphisms, a sequence of continuous positive endomorphisms) satisfying

$$\lim_{i \in I} L_i(y) = y, \qquad \text{for all} \quad y \in H$$

and

$$\{L_i(x - y_0) : i \in I\} \subset \hat{H}_{x-y_0} + E_+ = \hat{H}_x - y_0 + E_+ .$$

Setting $T_i := S \circ L_i$ for each $i \in I$ we deduce

$$\lim_{i \in I} T_i(y) = S(y) \qquad \text{for all} \quad y \in H.$$

Since $x - y_0 \in \overline{H}_S^{\mathscr{P}}$ (resp. $x - y_0 \in \overline{H}_S^{\mathscr{P}'}$, $x - y_0 \in \overline{H}_S^{\mathscr{P}_c}$, $x - y_0 \in \overline{H}_S^{\mathscr{P}_c'}$) it follows that $\lim_{i\in I} T_i(x - y_0) = S(x) - S(y_0)$. Moreover, for each $i \in I$ there is an element $y_i \in \hat{H}_x$ satisfying $T_i(x - y_0) \geqslant S(y_i) - S(y_0) \geqslant S(x) - S(y_0)$. Since F is a Hausdorff topological vector lattice this implies that $\lim_{i\in I} S(y_i) = S(x)$ and, consequently, $\lim_{y \in \hat{H}_x} S(y) = S(x)$. Replacing x by $-x$ we conclude $\lim_{y\in \check{H}^x} S(y) = S(x)$ which proves that x is (H, S)-affine.

Remarks 2.2. (i) If E is a locally convex Hausdorff topological vector lattice then the continuous positive linear forms separate points. Hence in this case we have $\mathscr{A}_S(H) = \overline{H}_S^{\mathscr{P}} = \overline{H}_S^{\mathscr{P}_c}$.

(ii) An example of Scheffold [9] shows that the equality $\overline{H}_S^{\mathscr{P}} = \overline{H}_S^{\mathscr{P}'}$ fails in general. In view of the applications H will often be finite dimensional or at most of countable algebraic dimension. By 2.1 the sets $\overline{H}_S^{\mathscr{P}}$ and $\overline{H}_S^{\mathscr{P}'}$ coincide under this assumption. Moreover, in the setting of Scheffold's example, $\overline{H}_S^{\mathscr{P}} = \overline{H}_S^{\mathscr{P}'}$ even if H is only separable [2, 4.6].

(iii) The characterization of the (H, S)-affine elements is not difficult, in general: If E is an M-space there are rather intuitive equivalent descriptions of the (H, S)-affine elements. Even, however, if E is not an M-space the following proposition may be helpful in determining the (H, S)-affine elements.

PROPOSITION 2.3. *Suppose that F is a locally convex Hausdorff vector lattice. Then an element $x \in E$ is (H, S)-affine iff* $\lim_{y\in\hat{H}_x} l(S(y)) = l(S(x)) = \lim_{y\in\check{H}^x} l(S(y))$ *for all $l \in F_+{}^0$, where $F_+{}^0$ is the polar of the positive cone F_+ in F*[5].

Proof. Since, for each $x \in E$, $(S(y))_{y\in\hat{H}_x}$, $(S(y))_{y\in\check{H}^x}$ are monotonic nets, these nets converge in F iff they are weakly convergent, and the respective limits coincide [5, p. 91].

EXAMPLE 2.4. Let $p \in [1, +\infty[$, and suppose that $(X, \mathfrak{A}, \mu)$ is a finite measure space. Then $E = F = L^p(\mu)$ is super Dedekind complete in the terminology of Luxemburg and Zaanen [6, p. 126]. Hence, for each $f \in L^p(\mu)$,

[5] $F_+{}^0$ is the cone of all continuous positive linear forms on F.

$\hat{f} := \inf \hat{H}_f$ and $\check{f} := \sup \check{H}^f$ are well-defined elements of $L^p(\mu)$ whenever $H_f \neq \varnothing$ and $H^f \neq \varnothing$. If I is the identity operator on $L^p(\mu)$ we obtain $\overline{H}_I^{\mathscr{P}} = \overline{H}_I^{\mathscr{P}_c} = \mathscr{A}_I(H) = \{f \in L^p(\mu) : H_f \neq \varnothing, H^f \neq \varnothing \text{ and } \hat{f} = \check{f}\}$.

Proof. It remains to show the last equality. To do this we use Proposition 2.3: Since each continuous positive linear form on $L^p(\mu)$ can be represented by an element of $L^q(\mu)_+$, where $1/q + 1/p = 1$, $q \in \mathbf{R}$, according to the Riesz representation theorem, we obtain:

$$\mathscr{A}_I(H) = \left\{ f \in L^p(\mu) : \inf_{h \in \hat{H}_f} \int hg \, d\mu = \sup_{h \in \check{H}^f} \int hg \, d\mu \quad \text{for all } g \in L^q(\mu)_+ \right\}.$$

From the order-separability of $L^p(\mu)$ it now follows that there is a decreasing sequence (h_n) in $\hat{H}_f$ such that $\inf_{n \in \mathbb{N}} h_n = \hat{f}\,(f \in E,\ H_f \neq \varnothing)$. Using the fact that $h \geqslant \hat{f}$ for all $h \in \hat{H}_f$ we conclude from this: ($g \in L^q(\mu)_+$ arbitrary)

$$\int \hat{f} g \, d\mu = \inf_{n \in \mathbb{N}} \int h_n g \, d\mu \geqslant \inf_{h \in \hat{H}_f} \int hg \, d\mu \geqslant \int \hat{f} g \, d\mu, \text{ i.e.,}$$

$$\int \hat{f} g \, d\mu = \inf_{h \in \hat{H}_f} \int hg \, d\mu, \text{ and, similarly, } \int \check{f} g \, d\mu = \sup_{h \in \check{H}^f} \int hg \, d\mu.$$

Consequently, f is (H, I)-affine iff $H_f \neq \varnothing$, $H^f \neq \varnothing$ and

$$\int (\hat{f} - \check{f})\, g \, d\mu = 0 \text{ for all } g \in L^q(\mu)$$

from which the assertion follows.

Remark 2.5. For each $z \in E$, let $\tilde{z} : E_+^0 \to \mathbf{R}$ be defined by $\tilde{z}(1) = 1(z)$ $(1 \in E_+^0)$. Then $\tilde{E} := \{\tilde{z} : z \in E\}$ is a linear lattice of continuous affine functions on E_+^0 under pointwise order. Proposition 2.3 shows that $\mathscr{A}_S(H)$ is the set of all $x \in E$ satisfying $H_x \neq \varnothing$, $H^x \neq \varnothing$ and

$$\sup\{\tilde{y} : y \in \check{H}^x\} = \tilde{x} = \inf\{\tilde{y} : y \in \hat{H}_x\} \text{ pointwise on } S^*(F_+^0),$$

where S^* is the adjoint operator of S.

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